Functional Analysis and Its Applications, Vol. 49, No. 1, pp. 50–56, 2015
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 49, No. 1, pp. 62–70, 2015

On Algebraic Functions Integrable in Finite Terms*

A. G. Khovanskii

To the memory of Vladimir Igorevich Arnold

Received April 30, 2013

Abstract. Liouville's theorem describes algebraic functions integrable in terms of generalized elementary functions. In many cases, algorithms based on this theorem make it possible to either evaluate an integral or prove that the integral cannot be "evaluated in finite terms." The results of the paper do not improve these algorithms but shed light on the arrangement of the 1-forms integrable in finite terms among all 1-forms on an algebraic curve.

Key words: Abelian integral, algebraic function, elementary function, solvability in finite terms.

1. Introduction. The integral of a rational function f of a complex variable z can be explicitly evaluated as

$$\int_{z_0}^{z} f(t)\,dt = f_0(z) + \sum_{1\leqslant i\leqslant k} \lambda_i \ln f_i(z), \tag{1}$$

where the f_i, $0 \leqslant i \leqslant k$, are rational functions and the λ_i are complex numbers. It is convenient to write relation (1) in the form

$$f\,dz = df_0 + \sum_{1\leqslant i\leqslant k} \lambda_i \frac{df_i}{f_i}\,.$$

When can the integral of an Abelian 1-form α be evaluated in finite terms? Thinking on this question, Abel laid the foundations of the theory of Abelian integrals. Liouville continued his work and found conditions under which the integral of a form α is a *generalized elementary function* (see Section 2).

Liouville's theorem. *The integral of a rational form α on an algebraic curve Γ is a generalized elementary function if and only if*

$$\alpha = df_0 + \sum_{1\leqslant i\leqslant k} \lambda_i \frac{df_i}{f_i}, \tag{2}$$

where the f_i, $0 \leqslant i \leqslant k$, are rational functions on Γ and the λ_i are complex numbers.

The proof of this theorem can be found, e.g., in the book [1]. In many cases, algorithms based on Liouville's theorem make it possible to either prove that the integral is nonelementary or evaluate it [2]. Liouville obtained a whole series of other results on the solvability and unsolvability of equations in finite terms. Afterwards, his pioneering works were generalized and translated into the language of differential algebra. An extensive bibliography on this question is contained in the survey [3].

We shall deal with the classical complex situation rather than with algebraic generalizations. Rational functions and 1-forms on a complex algebraic curve can be regarded as meromorphic functions and 1-forms on a compact Riemann surface. We shall use both terminologies.

There are two summands in (2), df_0 and $\sum \lambda_i d(f_i)/f_i$. The former is, obviously, contained in the subspace Ω_s of Ω of meromorphic 1-forms on Γ that consists of the forms all of whose residues vanish.

Consider the subspace $\Omega_l \subset \Omega$ consisting of the forms α having at most simple poles and such that $\int_\Gamma \alpha \wedge \beta = 0$ for any harmonic 1-form β. It turns out that Ω_l is complementary to Ω_s (i.e.,

*This work was supported in part by Canadian Grant no. 0GP0156833.

 0016–2663/15/4901–0050

$\Omega = \Omega_l + \Omega_s$) and contains logarithmic differentials, i.e., the second term $\sum \lambda_i d(f_i)/f_i$ belongs to Ω_l. This unexpected simple observation is the main result of the paper. As far as I know, it is new, although Liouville's theorem has been known for more than one and a half centuries, and it has been the object of numerous studies.

The referee of this paper noticed that Ω_l can be defined quite differently, as the complexification of the $\mathbb{R}$-linear space $\Omega_{\mathbb{R}}$ of all 1-forms with real periods and at most simple poles (see the end of Section 3). This observation clarifies the situation. Moreover, it relates Ω_l to the space $\Omega_{\mathbb{R}}$, which is encountered in many works (see [4]–[7]). I am grateful to the referee for this beautiful remark and for editing suggestions.

My interest in the solvability and unsolvability of equations in finite terms was aroused by V. I. Arnold. I am very much indebted to Vladimir Igorevich. I gratefully dedicate this work to his memory.

2. Generalized elementary functions on a Riemann surface. We begin with the definition of the class of generalized elementary functions of a complex variable z. This class is defined by specifying a set of *basic elementary functions* and a list of *admissible operations*. A function is said to be *generalized elementary* if it can be obtained from basic functions by applying admissible operations (a detailed discussion of this definition and related notions is contained in the book [1]).

Since we consider multivalued functions, we must specify what we mean. A multivalued analytic function is uniquely determined by its germ (i.e., a convergent Taylor series) at an arbitrary point; this is the set of all germs obtained by analytically continuing a given germ. For example, the superposition of multivalued functions is defined as follows. *A function F can be represented as the superposition of functions f and g* if

(i) there exist germs F_a and g_a of the functions F and g at some point $a \in \mathbb{C}$,

(ii) there exists a germ f_b of f at the point $b = g_a(a)$, and

(iii) $F_a = f_b \circ g_a$.

Thus, for each $k \in \mathbb{Z}$, the function $F(z) = z + 2k\pi i$ can be represented as the superposition $F = \ln \exp(z)$. (Indeed, let $b = \exp a$, and let $\ln_b$ be a germ at b of ln for which $\ln_b \circ \exp_a = z_a$. Each germ of the function ln at b has the form $\ln_b + 2k\pi i$. The function $z + 2k\pi i$ is the analytic continuation of the germ $(\ln_b + 2k\pi i) \circ \exp_a$.) Other operations on multivalued functions are defined in a similar way.

List of basic elementary functions:

- complex constants and the function z;
- $\exp z$, $\ln z$, and z^α with $\alpha \in \mathbb{C}$;
- $\sin z$, $\cos z$, and $\tan z$;
- $\arcsin z$, $\arccos z$, and $\arctan z$.

List of admissible operations:

- the arithmetic operations $+$, $-$, $\times$, and $:$;
- the superposition operation (to any functions f and g it assigns their composition $f \circ g$);
- the operation of taking a solution of an algebraic equation (to functions $f_1, \dots, f_n$ it assigns a function y such that $y^n + f_1 y^{n-1} + \dots + f_0 = 0$).

In essence, the functions in the list of basic elementary functions are studied in school mathematics and often present on calculator keyboards. To define generalized elementary functions, it would suffice to leave only the complex constants and the functions z, exp, and ln in this list. The remaining functions in the list are obtained from these by applying admissible operations (see [1, p. 8, Lemma 1.2]).

Let Γ be a compact Riemann surface on which a nonconstant meromorphic mapping $\pi \colon \Gamma \to \mathbb{C}$ is defined, and let f be a (multivalued) function on Γ. We say that f is a *generalized elementary function* on Γ if $f(\pi^{-1})$ is a generalized elementary function on $\mathbb{C}$. Let us show that this definition does not depend on the choice of the mapping π.

Lemma 1. *If $f(\pi^{-1})$ is a generalized elementary function, then so is $f(\pi_0^{-1})$ for any nonconstant meromorphic mapping $\pi_0 \colon \Gamma \to \mathbb{C}$.*

Proof. The function $u = \pi(\pi_0^{-1}(z))$ is algebraic. By assumption, $g(u) = f(\pi^{-1}(u))$ is a generalized elementary function. But $f(\pi_0^{-1})(z) = g(u(z))$. This completes the proof of the lemma.

3. Forms of logarithmic type. On the complex line $\mathbb{C}$, the function r^{-1}, where $r(a)$ is the distance from $a \in \mathbb{C}$ to the zero $0 \in \mathbb{C}$, is integrable in a neighborhood of 0. Therefore, a meromorphic function with at most simple poles on a compact domain $K \subset \mathbb{C}$ is a function of class L^1 on this domain K. Let Γ be a compact Riemann surface endowed with some Riemannian metric compatible with the complex structure. As seen from the above considerations, a meromorphic 1-form α on Γ with at most simple poles belongs to the class L^1. Let $A \subset \Gamma$ be a finite set containing the poles of α. The form α is closed on the domain $\Gamma \setminus A$, but as a current on the curve Γ it is not closed:

$$d\alpha = \frac{1}{2\pi i} \sum_{a \in A} \operatorname{Res} \alpha(a) \delta(a),$$

where $\delta(a)$ is the 2-current whose value at any smooth function ϕ on Γ equals $\phi(a)$. For any smooth 1-form β on Γ, the integral $\int_\Gamma \alpha \wedge \beta$ is defined. We say that a 1-form α is a *logarithmic differential* if there exists a rational function f such that $\alpha = df/f$. This paper is based on the following simple observation.

Theorem 2. *For any logarithmic differential $\alpha = df/f$ and any harmonic form β on a curve Γ,*

$$\int_\Gamma \alpha \wedge \beta = 0.$$

Proof. The harmonic form β decomposes into the sum of a holomorphic form ω_1 and an antiholomorphic form $\overline{\omega}_2$. For a holomorphic form ω_1, we have $\alpha \wedge \omega_1 \equiv 0$; therefore, $\int_\Gamma \alpha \wedge \omega_1 = 0$. Let us show that $\int_\Gamma \alpha \wedge \overline{\omega}_2 = 0$. Consider a ramified covering $\pi\colon \Gamma \to \mathbb{C}P^1 = \mathbb{C}^1 \cup \{\infty\}$ determined by the function f, i.e., a covering for which $f = z \circ \pi$, where $z\colon \mathbb{C}^1 \to \mathbb{C}^1$ is a coordinate on $\mathbb{C}^1$. For any form Φ on Γ, let $\operatorname{Trace}\Phi$ denote the trace of Φ under the projection π. According to Abel's theorem, we have $\operatorname{Trace}\omega_2 \equiv 0$; therefore, $\operatorname{Trace}\overline{\omega}_2 \equiv \overline{\operatorname{Trace}\omega_2} \equiv 0$. The trace $\operatorname{Trace}(\alpha \wedge \overline{\omega}_2)$ vanishes as well, because

$$\operatorname{Trace}(\alpha \wedge \overline{\omega}_2) = \operatorname{Trace}\left(\frac{d(z \circ \pi)}{z \circ \pi} \wedge \overline{\omega}_2\right) = \frac{d(z \circ \pi)}{z \circ \pi} \wedge \operatorname{Trace}\overline{\omega}_2 \equiv 0.$$

We have $\int_\Gamma \alpha \wedge \overline{\omega}_2 = \int_{\mathbb{C}P^1} \operatorname{Trace}(\alpha \wedge \overline{\omega}_2) = 0$. This completes the proof of the theorem.

We say that a meromorphic 1-form α on Γ is of *logarithmic type* if (1) all poles of α are simple; (2) for any harmonic form β, the integral $\int_\Gamma \alpha \wedge \beta$ vanishes.

Let Ω_l denote the space of forms of logarithmic type, and let $\Omega_{\ln}$ be the space of forms which can be represented as combinations $\sum_{1 \leqslant i \leqslant k} \lambda_i df_i/f_i$ of logarithmic differentials.

Corollary 3. *The inclusion $\Omega_{\ln} \subset \Omega_l$ holds.*

Theorem 4. *Let $\phi\colon \Gamma \to \mathbb{C}$ be a function taking nonzero values only on a finite set and such that $\sum_{a \in \Gamma} \phi(a) = 0$. Then there is a unique form $\alpha_\phi \in \Omega_l$ whose residue $\operatorname{Res}\alpha_\phi(a)$ at each point a equals $\phi(a)$.*

Proof. By the Riemann–Roch theorem, there exists a form α_1 having only simple poles and such that $\operatorname{Res}\alpha_1 = \phi$, where $\operatorname{Res}\alpha_1 : \Gamma \to \mathbb{C}$ is the function whose value at each point $a \in \Gamma$ is the residue of the form α_1 at a. (The form α_1 is unique up to the addition of a holomorphic form.) Consider the linear function F_{α_1} on the space of antiholomorphic forms $\overline{\omega}$ defined by $F_{\alpha_1}(\overline{\omega}) = \int_\Gamma \alpha_1 \wedge \omega$. As any linear function on the space of antiholomorphic forms, F_{α_1} can be represented in the form $F_{\alpha_1}(\overline{\omega}) = \int_\Gamma \omega_1 \wedge \overline{\omega}$, where ω_1 is a holomorphic form, which is uniquely determined by this condition. It remains to set $\alpha_\phi = \alpha_1 - \omega_1$. This completes the proof of the theorem.

Corollary 5. *The dimension of the space $\Omega_l(A) \subset \Omega_l$ consisting of forms whose poles belong to a finite set $A \subset \Gamma$ equals $\#A - 1$.*

Remark. The following argument due to the referee relates Ω_l to the $\mathbb{R}$-linear space $\Omega_{\mathbb{R}}$ of forms with at most simple poles all of whose periods are real. If $\alpha \in \Omega_{\mathbb{R}}$ and ω are holomorphic forms, then the integrals $\int_\Gamma \alpha \wedge \omega$ and $\int_\Gamma \alpha \wedge \overline{\omega}$ are complex conjugate (because the integral of α over any cycle is real and the integrals of the forms ω and $\overline{\omega}$ over any 1-chain with coefficients in the field $\mathbb{R}$ are conjugate). But $\int_\Gamma \alpha \wedge \omega = 0$; therefore, $\int_\Gamma \alpha \wedge \overline{\omega} = 0$. Thus, $\Omega_\mathbb{R} \subset \Omega_l$ and $\Omega_\mathbb{R} + i\Omega_\mathbb{R} \subset \Omega_l$. Any function ϕ (see Theorem 4) can be represented as a residue of some form from $\Omega_\mathbb{R} + i\Omega_\mathbb{R}$; hence $\Omega_\mathbb{R} + i\Omega_\mathbb{R} = \Omega_l$. Obviously, $df/f \in i\Omega_\mathbb{R}$ for any meromorphic function f. This argument provides an alternative proof of the results of Section 3. Further information on the space $\Omega_\mathbb{R}$ and its diverse applications can be found in [4]–[7].

4. Forms of the second kind and a decomposition theorem. We say that a meromorphic 1-form is a *form of the second kind* if its residue at each point vanishes (in our terminology, all holomorphic forms are of the second kind).

Let Ω_d denote the space of exact forms, i.e., forms $\alpha = df$, where f is a rational function. Obviously, *any exact form is of the second kind*, i.e., $\Omega_d \subset \Omega_s$.

Theorem 6 (on decomposition). *The space Ω of meromorphic 1-forms on Γ decomposes into the direct sum of the subspaces Ω_s and Ω_l.*

Proof. Given $\alpha \in \Omega$, we set $\phi = \operatorname{Res} \alpha$ and define α_l and α_s by $\alpha_l = \alpha_\phi$ (see the proof of Theorem 4) and $\alpha_s = \alpha - \alpha_l$. Clearly, $\alpha = \alpha_s + \alpha_l$, $\alpha_s \in \Omega_s$, and $\alpha_l \in \Omega_l$. The spaces Ω_s and Ω_l do not intersect. This proves the theorem.

The decomposition of forms provided by Theorem 6 agrees with Liouville's theorem.

Theorem 7. *Let $\alpha = \alpha_s + \alpha_l$ be the decomposition of the form α from Theorem 6. The antiderivative of the form α is a generalized elementary function if and only if $\alpha_s \in \Omega_d$ is an exact form and $\alpha_l \in \Omega_{\ln}$ is a linear combination of logarithmic differentials.*

Proof. According to Liouville's theorem, the antiderivative of α is a generalized elementary function if and only if α can be represented in the form (2). Moreover, the term df_0 belongs to $\Omega_d \subset \Omega_s$ and $\sum \lambda_i df_i/f_i$ belongs to $\Omega_{\ln} \subset \Omega_l$ (see Corollary 3). This proves the theorem.

To Liouville's theorem the following two problems are related.

Problem 1. Does a given form $\alpha \in \Omega_s$ belong to the subspace Ω_d?

Problem 2. Does a given form $\alpha \in \Omega_l$ belong to the subspace $\Omega_{\ln}$?

Problem 1 is discussed in Section 5 and Problem 2, in Section 6.

5. Exact rational forms. Let us show that the codimension of the subspace $\Omega_d \subset \Omega_s$ equals twice the genus g of the curve Γ.

Theorem 8. *The space Ω_s/Ω_d is isomorphic to the first de Rham cohomology space $H^1(\Gamma, \mathbb{C})$ of the curve Γ. In particular, $\dim_\mathbb{C} \Omega_s/\Omega_d = 2g$.*

Proof. Each form $\alpha_s \in \Omega_s$ determines a one-dimensional de Rham cohomology class of the curve Γ. Indeed, if a cycle γ bounds a domain on Γ and passes through no poles of α_s, then $\int_\gamma \alpha_s = 0$, because all residues of the form α_s vanish. To complete the proof, it remains to apply Lemma 9 proved below.

Let $A \subset \Gamma$ be a nonempty finite set, and let $\Omega(A)$ be the space of rational forms regular in $\Gamma \setminus A$. We use $\Omega_s(A)$ and $\Omega_d(A)$ to denote the intersections of $\Omega(A)$ with the spaces Ω_s and Ω_d.

Lemma 9. *For any cohomology class $h \in H^1(\Gamma, \mathbb{C})$, there exists a form $\alpha \in \Omega_s(A)$ representing the class h.*

Proof. The curve $X = \Gamma \setminus A$ has the structure of a one-dimensional smooth affine algebraic manifold, with respect to which all meromorphic functions and 1-forms with poles in A on X are regular functions and 1-forms. According to the de Rham–Grothendieck theorem, any one-dimensional cohomology class of the affine curve X can be represented by a form from $\Omega(A)$. The form α representing h has zero residues and, therefore, belongs to $\Omega_s(A)$.

Let $D = \sum_{a_i \in A} m_i a_i$ be a divisor supported on A whose coefficients satisfy the inequalities $m_i \geqslant 2$, and let $\Omega_s[D]$ be the space of forms $\beta \in \Omega_s(A)$ for which $(\beta) \leqslant D$; we set $\Omega_d[D] =$

$\Omega_s[D] \cap \Omega_d$. Let us calculate the codimension of the subspace $\Omega_d[D] \subset \Omega_s[D]$ for which Problem 1 is solvable. We use the following notation:

D' is the divisor defined by $D' = D - \sum_{a_i \in A} a_i$;

$\mathscr{L}(D')$ is the space of functions f such that $D' + (f) \geqslant 0$;

$l(D') = \dim_{\mathbb{C}} \mathscr{L}(D')$;

$\mathscr{I}(D')$ is the space of forms β such that $(\beta) \geqslant D'$;

$\mu(D') = \dim_{\mathbb{C}} \mathscr{I}(D')$.

Statement 10. *The codimension of the subspace $\Omega_d[D]$ in $\Omega_s[D]$ equals $2g - \mu(D')$.*

Proof. By the Riemann–Roch theorem, we have

(i) $\dim_{\mathbb{C}} \Omega_s[D] = (\deg D + g - 1) - (\#A - 1) = \deg D' + g$;

(ii) $l(D') = \deg D' - g + 1 + \mu(D')$.

The dimension of the space of differentials of functions belonging to $\mathscr{L}(D')$ equals $l(D') - 1$ (differentiation takes constants to zero). Therefore, the codimension of the subspace $\Omega_d[D] \subset \Omega_s[D]$ equals $\dim_{\mathbb{C}} \Omega_s[D] - (l(D') - 1) = 2g - \mu(D')$. This proves the statement.

Note that if $\deg D' > 2g-2$, then $\mu(D') = 0$, so that Statement 10 is a refinement of Theorem 8.

Let us comment on Statement 10. Near each point $a \in A$ we fix a local coordinate z so that $z(a) = 0$. Suppose that a form $\alpha \in \Omega_s[D]$ can be represented near a point $a \in A$ as

$$\alpha = \left(\frac{c_k}{z^k} + \cdots + \frac{c_2}{z^2} + \varphi \right) dz,$$

where φ is a germ of a holomorphic function at a. The germ

$$I_a = \frac{(-k+1)c_k}{z^{k-1}} + \cdots + \frac{-c_2}{z}$$

is the principal part of the integral of α near the point a: the germ $\alpha - dI_a$ is holomorphic near a.

Theorem 11 (on Problem 1). *Problem 1 for a form $\alpha \in \Omega_s[D]$ is solvable if and only if $\sum_{a \in A} \operatorname{Res}(I_a \omega) = 0$ for the sets of principal parts I_a of the integral of α and any form ω holomorphic on Γ.*

Proof. By the Riemann–Roch theorem, a meromorphic function with a given set of principal parts I_a exists if and only if $\sum_{a \in A} \operatorname{Res}(I_a \omega) = 0$ for any holomorphic form ω.

Statement 10 is proved by counting the independent conditions imposed on the sets of principal parts I_a by Theorem 11.

Remark. How can a function on a given algebraic curve with a given set of principal parts be found explicitly, provided that such a function exists? Theorem 11 is not very helpful in answering this question. There have been developed effective methods for finding such a function (of course, they depend on the way of specifying both the curve and the principal parts of the function) (see [2]).

6. Logarithmic differentials. A form $\alpha \in \Omega_{ln}$ admits various representations

$$\alpha = \sum_{1 \leqslant i \leqslant k} \lambda_i \frac{df_i}{f_i}, \tag{3}$$

where the f_i are rational functions on Γ. Let A be the set of poles of α. The following simple statement is well known (see, e.g., [1, p. 17, Lemma 1.10]).

Statement 12. *If representation* (3) *contains the least number k of summands, then the λ_i are independent over $\mathbb{Q}$ and the supports of the divisors of all functions f_i are contained in A.*

Now we give a few definitions and introduce notation. Let $A \subset \Gamma$ be a finite set. Let $J_0(A)$ denote the set of functions $\phi\colon A \to \mathbb{Z}$ for which the divisor $D_\phi = \sum \phi(a)a$ is principal. This set $J_0(A)$ is an additive group. We use $\mathscr{D}(A)$ to denote the complex linear space of functions on A generated by the group $J_0(A)$. The dimension $\dim_{\mathbb{C}} \mathscr{D}(A)$ is called the *rank* of A and denoted by $r(A)$. We denote the intersections of the spaces $\Omega_{\ln}$ and Ω_l with $\Omega(A)$ by $\Omega_{\ln}(A)$ and $\Omega_l(A)$, respectively.

Theorem 13 (on Problem 2). *A form $\alpha \in \Omega_l(A)$ belongs to $\Omega_{\ln}(A)$ if and only if the function $\operatorname{Res}\alpha\colon A \to \mathbb{C}$ belongs to $\mathscr{D}(A)$. Therefore, $\dim_{\mathbb{C}} \Omega_{\ln}(A) = r(A)$.*

Proof. Let $\operatorname{Res}\alpha \in \mathscr{D}(A)$, and let $\operatorname{Res}\alpha = \sum \lambda_i \operatorname{ord} f_i$, where the f_i are rational functions whose divisors (f_i) are supported in A and $\operatorname{ord} f_i$ is the function whose value at each point $a \in \Gamma$ equals the order of f_i at a. Then $\alpha = \sum \lambda_i df_i/f_i$. Conversely, if $\alpha \in \Omega_l(A)$ belongs to the space $\Omega_{\ln}$, then, by Lemma 9, $\alpha \in \Omega_{\ln}(A)$; therefore, $\operatorname{Res}\alpha \in \mathscr{D}(A)$.

Remark. How can an explicit representation of a form on an algebraic curve as a linear combination of logarithmic differentials be obtained, provided that such a representation exists? Theorem 13 is not very helpful in answering this question. There have been developed methods making it possible to find such a linear combination in many cases (of course, these methods depend on the way of specifying both the form and the curve) (see [2]).

7. Codimension of forms integrable in finite terms. Theorems 8 and 13 involve calculating the codimension of the space $\Omega_e(A) \subset \Omega(A)$ of forms which have a given set A of poles on Γ and whose antiderivatives are generalized elementary functions. The space $\Omega_e(\emptyset)$ contains only the form $\alpha \equiv 0$. In what follows, we assume that $A \neq \emptyset$.

Corollary 14 (on codimensions). *The codimension of the subspace*

(i) $\Omega_d(A)$ *in the space* $\Omega_s(A)$ *equals* $2g$, *where* g *is the genus of the curve* Γ;

(ii) $\Omega_{\ln}(A)$ *in the space* $\Omega_l(A)$ *equals* $\#(A) - r(A) - 1$;

(iii) $\Omega_e(A)$ *in the space* $\Omega(A)$ *equals* $2g + \#(A) - r(A) - 1$.

A set $A \subset \Gamma$ with $\#A = k$ can be treated as a point in the kth symmetric power $\Gamma^{(k)}$ of the curve Γ. Let $\Sigma^{(k)} \subset \Gamma^{(k)}$ be the set of all $A \subset \Gamma$ with $\#A = k$ and $r(A) > 0$.

Statement 15. *For a curve Γ of positive genus, the set $\Sigma^{(k)}$ has measure zero in $\Gamma^{(k)}$.*

Proof. If $r(A) > 0$, then there exists a principal divisor $D = \sum k_i a_i$, where $a_i \in A$ and $\sum k_i = 0$. In this case, the points in A satisfy the nontrivial relation $\sum k_i a_i = 0$ on the Jacobian of Γ. This implies Statement 15.

Corollary 16. *On a curve Γ of genus $g > 0$, for almost every nonempty set A, the codimension of $\Omega_e(A)$ in $\Omega(A)$ equals $2g + \#A - 1 = \dim H^1(\Gamma \setminus A, \mathbb{C})$.*

Below we give examples of nonempty sets A on curves of arbitrarily large genus for which the upper bound of $r(A)$, which equals $\#A - 1$, is attained. We begin with the case of curves of genus 1.

Let (Γ, a_0) be a curve of genus 1 with base point a_0. We say that a point $a \in \Gamma$ is *of finite order* if, for some positive integer k, the divisor $ka - ka_0$ is principal. The points of finite order are dense in Γ.

Example 1. On a curve of genus 1, any finite nonempty set A of points of finite order has rank $r(A) = \#A - 1$.

Thus, on a curve of genus 1, *the antiderivative of any form $\alpha \in \Omega_l(A)$, where A consists of points of finite order, is a linear combination of logarithms of rational functions on Γ. The codimension of the subspace $\Omega_e(A)$ in $\Omega(A)$ equals* $\dim H^1(\Gamma, \mathbb{C}) = 2$.

Let R be a rational function of degree k in a complex variable z with simple zeros and poles. Given $m > 0$, consider the algebraic function $y(z)$ defined by $y^m = R(z)$ and the Riemann surface $\pi\colon \Gamma \to \mathbb{C}P^1$ of this function. According to the Riemann–Hurwitz formula, the genus of the surface Γ equals $(k-1)(m-1)+1$ and can be arbitrarily large.

Example 2. Let Σ be the set of zeros and poles of the function R, and let $A = \pi^{-1}(\Sigma)$. Any divisor of degree zero supported in A becomes a principal divisor $(\pi^* f)$ after multiplication by m, where f is some rational function of the variable z. Therefore, we have $r(A) = \#A - 1$.

Thus, under the assumptions of Example 2, *the antiderivative of any form $\alpha \in \Omega_l(A)$ is a linear combination of logarithms of rational functions on Γ, and the codimension of the subspace $\Omega_e(A)$ in $\Omega(A)$ equals* $\dim H^1(\Gamma, \mathbb{C})$.

References

[1] A. G. Khovanskii, Topological Galois Theory. Solvability and Unsolvability of Equations in Finite Terms, Springer Monographs in Mathematics, Springer-Verlag, Berlin–Heidelberg, 2014.

[2] J. Davenport, On the Integration of Algebraic Functions, Lecture Notes in Computer Science, vol. 102, Springer-Verlag, Berlin–New York, 1981.

[3] M. F. Singer, "Formal solutions of differential equations," J. Symbolic Comput., **10**:1 (1990), 59–94.

[4] S. Grushevsky and I. Krichever, "The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces and the geometry of the moduli space of pointed Riemann surfaces," in: Geometry of Riemann Surfaces and Their Moduli Spaces. Surveys in Differential Geometry, vol. XIV, International Press, Somerville, MA, 2009, 111–129.

[5] I. Krichever and D. Zakharov, "A note on critical points of soliton equations," Anal. Math. Phys., **1**:1 (2011), 15–35.

[6] S. Grushevsky and I. Krichever, Foliations on the Moduli Space of Curves, Vanishing in Cohomology, and Calogero–Moser Curves, `http://arxiv.org/abs/1108.4211`.

[7] I. M. Krichever, "Real normalized differentials and Arbarello's conjecture," Funkts. Anal. Prilozhen., **46**:2 (2012), 37–51; English transl.: Functional Anal. Appl., **46**:2 (2012), 110–120.

Institute for System Analysis, Russian Academy of Sciences
Independent University of Moscow
The University of Toronto, Canada
e-mail: askold@math.toronto.edu

Translated by O. V. Sipacheva